The double angle formulae

This unit looks at trigonometric formulae known as the **double angle formulae**. They are called this because they involve trigonometric functions of double angles, i.e. $\sin 2A$, $\cos 2A$ and $\tan 2A$.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- derive the double angle formulae from the addition formulae
- write the formula for $\cos 2A$ in alternative forms
- use the formulae to write trigonometric expressions in different forms
- use the formulae in the solution of trigonometric equations

Contents

1. Introduction

This unit looks at trigonometric formulae known as the **double angle formulae**. They are called this because they involve trigonometric functions of double angles, i.e. $\sin 2A$, $\cos 2A$ and $\tan 2A$.

2. The double angle formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$

We start by recalling the addition formulae which have already been described in the unit of the same name.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

We consider what happens if we let B equal to A. Then the first of these formulae becomes:

$$\sin(A+A) = \sin A\cos A + \cos A\sin A$$

so that

$$\sin 2A = 2\sin A\cos A$$

This is our first **double-angle formula**, so called because we are doubling the angle (as in 2A).

Similarly, if we put B equal to A in the second addition formula we have

$$\cos(A+A) = \cos A\cos A - \sin A\sin A$$

so that

$$\cos 2A = \cos^2 A - \sin^2 A$$

and this is our second double angle formula.

Similarly

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

so that

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

These three double angle formulae should be learnt.

Key Point

$$\sin 2A = 2\sin A\cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

3. The formula $\cos 2A = \cos^2 A - \sin^2 A$

We now examine this formula more closely.
We know from an important trigonometric identity that

$$\cos^2 A + \sin^2 A = 1$$

so that by rearrangement

$$\sin^2 A = 1 - \cos^2 A.$$

So using this result we can replace the term $\sin^2 A$ in the double angle formula. This gives

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1 - \cos^2 A) \\ &= 2\cos^2 A - 1 \end{aligned}$$

This is another double angle formula for $\cos 2A$.

Alternatively we could replace the term $\cos^2 A$ by $1 - \sin^2 A$ which gives rise to:

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= (1 - \sin^2 A) - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}$$

which is yet a third form.

Key Point

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

4. Finding $\sin 3x$ in terms of $\sin x$

Example

Consider the expression $\sin 3x$. We will use the addition formulae and double angle formulae to write this in a different form using only terms involving $\sin x$ and its powers.

We begin by thinking of $3x$ as $2x + x$ and then using an addition formula:

$$
\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x && \text{using the first addition formula} \\
&= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x && \text{using the double angle formula} \\
& && \cos 2x = 1 - 2\sin^2 x \\
&= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x && \text{from the identity } \cos^2 x + \sin^2 x = 1 \\
&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x
\end{aligned}
$$

We have derived another identity

$$\sin 3x = 3\sin x - 4\sin^3 x$$

Note that by using these formulae we have written $\sin 3x$ in terms of $\sin x$ (and its powers). You could carry out a similar exercise to write $\cos 3x$ in terms of $\cos x$.

5. Using the formulae to solve an equation

Example

Suppose we wish to solve the equation $\cos 2x = \sin x$, for values of x in the interval $-\pi \le x < \pi$.

We would like to try to write this equation so that it involves just one trigonometric function, in this case $\sin x$. To do this we will use the double angle formula

$$\cos 2x = 1 - 2\sin^2 x$$

The given equation becomes

$$1 - 2\sin^2 x = \sin x$$

which can be rewritten as

$$0 = 2\sin^2 x + \sin x - 1$$

This is a quadratic equation in the variable $\sin x$. It factorises as follows:

$$0 = (2\sin x - 1)(\sin x + 1)$$

It follows that one or both of these brackets must be zero:

$$2\sin x - 1 = 0 \qquad \text{or} \qquad \sin x + 1 = 0$$

so that

$$\sin x = \frac{1}{2} \qquad \text{or} \qquad \sin x = -1$$

We can solve these two equations by referring to the graph of $\sin x$ over the interval $-\pi \le x < \pi$ which is shown in Figure 1.

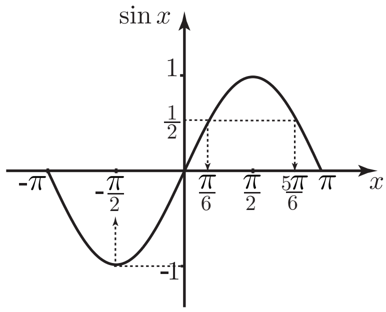

Figure 1. A graph of $\sin x$ over the interval $-\pi \le x < \pi$.

From the graph we see that the angle whose sine is -1 is $-\dfrac{\pi}{2}$. The angle whose sine is $\dfrac{1}{2}$ is a standard result, namely $\dfrac{\pi}{6}$, or $30°$. Using the graph, and making use of symmetry we note there is another solution at $x = \dfrac{5\pi}{6}$. So, in summary, the solutions are

$$x = \frac{\pi}{6}, \quad \frac{5\pi}{6} \quad \text{and} \quad -\frac{\pi}{2}$$

Example

Suppose we wish to solve the equation

$$\sin 2x = \sin x \qquad \pi \leq x < \pi$$

In this case we will use the double angle formulae $\sin 2x = 2\sin x \cos x$.
This gives

$$2\sin x \cos x = \sin x$$

We rearrange this and factorise as follows:

$$\begin{aligned} 2\sin x \cos x - \sin x &= 0 \\ \sin x(2\cos x - 1) &= 0 \end{aligned}$$

from which

$$\sin x = 0 \qquad \text{or} \qquad 2\cos x - 1 = 0$$

We have reduced the given equation to two simpler equations. We deal first with $\sin x = 0$. By referring to the graph of $\sin x$ in Figure 1 we see that the two required solutions are $x = -\pi$ and $x = 0$. The potential solution at $x = \pi$ is excluded because it is outside the interval specified in the original question.

The equation $2\cos x - 1 = 0$ gives $\cos x = \frac{1}{2}$. The angle whose cosine is $\frac{1}{2}$ is $60°$ or $\frac{\pi}{3}$, another standard result. By referring to the graph of $\cos x$ shown in Figure 2 we deduce that the solutions are $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

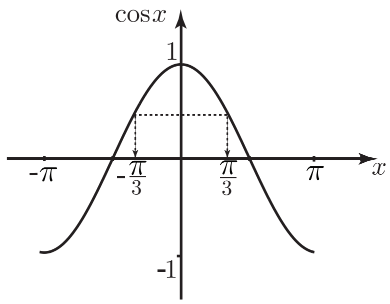

Figure 2. A graph of $\cos x$ over the interval $-\pi \leq x < \pi$.

Exercises

1. Verify the three double angle formulae (for $\sin 2A$, $\cos 2A$, $\tan 2A$) for the cases $A = 30^{o}$ and $A = 45^{o}$.

2. By writing $\cos(3x) = \cos(2x + x)$ determine a formula for $\cos(3x)$ in terms of $\cos x$.

3. Determine a formula for $\cos(4x)$ in terms of $\cos x$.

4. Solve the equation $\sin 2x = \cos x$ for $-\pi \le x < \pi$.

5. Solve the equation $\cos 2x = \cos x$ for $0 \le x < \pi$

Answers

2. $4\cos^3 x - 3\cos x$

3. $8\cos^4 x - 8\cos^2 x + 1$

4. $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

5. 0 and $\frac{2\pi}{3}$